A Greedy Approach for Dynamic Control of Diffusion Processes in Networks *Supplementary Material*

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I. INTRODUCTION

This document provides the supplementary technical material for the mentioned paper which is published in the annual *IEEE International Conference on Tools with Artificial Intelligence (ICTAI)*, 2015. This document, along with the release of the software package developed by Kevin Scaman and Argyris Kalogeratos for conducting the simulations for this work, are available at: `http://kalogeratos.com/material/lrie-dra/`.

II. APPENDIX

The second order derivative of the number of infected nodes is computed as the sum of three derivatives:

$$\begin{aligned} \frac{d^2}{dt^2}\mathbb{E}[N_I(t)] = & -\delta \frac{d}{dt}\mathbb{E}[N_I(t)] \\ & -\rho \frac{d}{dt}\mathbb{E}[X(t)^\top R(t)] \\ & +\beta \frac{d}{dt}\mathbb{E}[X(t)^\top A\overline{X}(t)]. \end{aligned} \tag{2.1}$$

In the following, we show that only the third derivative $\frac{d}{dt}\mathbb{E}[X(t)^\top A\overline{X}(t)]$ depends on $R(t)$ when $R(t)$ already minimizes $\frac{d}{dt}\mathbb{E}[N_I(t)]$. First, Eq. 4.9 shows that $\frac{d}{dt}\mathbb{E}[N_I(t)]$ does not depend on $R(t)$ since $X(t)^\top R(t) = \min(b_{tot}, N_I(t))$. Second, let $H(t) = \min(b_{tot}, N_I(t))$, then $\frac{d}{dt}\mathbb{E}[X(t)^\top R(t)]$ can be computed as follows:

$$\frac{d}{dt}\mathbb{E}[X(t)^\top R(t)] = \lim_{\Delta t \to 0} \frac{\mathbb{E}[H(t+\Delta t)] - \mathbb{E}[H(t)]}{\Delta t}. \tag{2.2}$$

Let Δt be a sufficiently small time interval. Three scenarios are possible:

- either $N_I(t) > b_{tot}$ and, during $t' \in [t, t+\Delta t]$, $H(t')$ is stationary (since $N_I(t')$ can at most increase or decrease by one),
- either $N_I(t) < b_{tot}$ and $H(t') = N_I(t')$ during Δt,
- or the last possibility is $N_I(t) = b_{tot}$, and in this case we only have to consider the case where $N_I(t+\Delta t) = b_{tot} - 1$ ($H(t')$ will not change if $N_I(t)$ increases).

Let $\mathbb{1}_{\{c\}} \in \mathbb{R}^{\mathbb{N}}$ be a vector with unit values at dimensions where a certain *condition* c is true, and $\overline{X}(t) = \mathbb{1} - X(t)$ be the vector indicating the healthy nodes of the network. We can then write:

$$\begin{aligned} & \mathbb{E}[H(t+\Delta t)|X(t)] \\ & \quad = \mathbb{1}_{\{N_I(t) > b_{tot}\}} b_{tot} \\ & \quad\quad + \mathbb{1}_{\{N_I(t) = b_{tot}\}} [b_{tot} - \delta N_I(t)\Delta t - \rho X(t)^\top R(t)\Delta t] \\ & \quad\quad + \mathbb{1}_{\{N_I(t) < b_{tot}\}} [N_I(t) - \delta N_I(t)\Delta t - \rho X(t)^\top R(t)\Delta t \\ & \quad\quad\quad + \beta X(t)^\top A\overline{X}(t)\Delta t] \\ & \quad\quad\quad + o(\Delta t) \\ & \quad = H(t) \\ & \quad\quad - \mathbb{1}_{\{N_I(t) \leq b_{tot}\}} [\delta N_I(t) + \rho H(t)]\Delta t \\ & \quad\quad + \mathbb{1}_{\{N_I(t) < b_{tot}\}} \beta X(t)^\top A\overline{X}(t)\Delta t \\ & \quad\quad + o(\Delta t). \end{aligned} \tag{2.3}$$

We thus have:

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[X(t)^\top R(t)] & = \lim_{\Delta t \to 0} \frac{\mathbb{E}[\mathbb{E}[H(t+\Delta t)|X(t)]] - \mathbb{E}[H(t)]}{\Delta t} \\ & = -\mathbb{E}[\mathbb{1}_{\{N_I(t) \leq b_{tot}\}}(\delta N_I(t) + \rho H(t))] \\ & \quad + \mathbb{E}[\mathbb{1}_{\{N_I(t) < b_{tot}\}} \beta X(t)^\top A\overline{X}(t)], \end{aligned} \tag{2.4}$$

which does not depend on $R(t)$.

Finally, $\frac{d}{dt}\mathbb{E}[X(t)^\top A\overline{X}(t)]$ is the only term depending on $R(t)$ (using Eq. 4.7 and Eq. 4.8):

$$\begin{aligned} & \frac{d}{dt}\mathbb{E}[X(t)^\top A\overline{X}(t)] \\ & \quad = \sum_{i,j} A_{ij} \frac{d}{dt}\mathbb{E}[X_i(t)\overline{X}_j(t)] \\ & \quad = \sum_{i,j} A_{ij} (\frac{d}{dt}\mathbb{E}[X_i(t)] - \frac{d}{dt}\mathbb{E}[X_i(t)X_j(t)]) \\ & \quad = -\delta \sum_{i,j} A_{ij}\mathbb{E}[X_i(t)] \\ & \quad\quad -\rho \sum_{i,j} A_{ij}\mathbb{E}[X_i(t)R_i(t)] \\ & \quad\quad +\beta \sum_{i,j,k} A_{ij}A_{ki}\mathbb{E}[\overline{X}_i(t)X_k(t)] \\ & \quad\quad +2\delta \sum_{i,j} A_{ij}\mathbb{E}[X_i(t)X_j(t)] \\ & \quad\quad +\rho \sum_{i,j} A_{ij}\mathbb{E}[X_i(t)X_j(t)(R_i(t) + R_j(t))] \\ & \quad\quad -\beta \sum_{i,j,k} A_{ij}A_{ki}\mathbb{E}[\overline{X}_i(t)X_j(t)X_k(t)] \\ & \quad\quad -\beta \sum_{i,j,k} A_{ij}A_{kj}\mathbb{E}[X_i(t)\overline{X}_j(t)X_k(t)]. \end{aligned} \tag{2.5}$$

This equation is simplified by the fact that, in order to minimize $\frac{d}{dt}\mathbb{E}[N_I(t)]$, resources are only given to infected nodes, which implies $X_i(t)R_i(t) = R_i(t)$. We can thus rewrite this derivative as:

$$
\begin{aligned}
&\tfrac{d}{dt}\mathbb{E}[X(t)^\top A\overline{X}(t)] \\
&\quad= -\rho \textstyle\sum_{i,j} A_{ij}\mathbb{E}[R_i(t)] \\
&\qquad+\rho \textstyle\sum_{i,j} A_{ij}\mathbb{E}[X_j(t)R_i(t) + X_i(t)R_j(t)] \\
&\qquad+\Xi(t) \\
&\quad= -\rho\mathbb{E}[\mathbb{1}^\top A^\top R(t)] \\
&\qquad+\rho\mathbb{E}[X(t)^\top A^\top R(t) + X(t)^\top AR(t)] \\
&\qquad+\Xi(t) \\
&\quad= -\rho\mathbb{E}[\{A\overline{X}(t) - A^\top X(t)\}^\top R(t)] \\
&\qquad+\Xi(t),
\end{aligned}
\tag{2.6}
$$

where $\Xi(t)$ is independent of $R(t)$. This leads to the second order derivative of $\mathbb{E}[N_I(t)]$ given in Eq. 4.11.